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Littlewood-Paley theorem for Schrodinger operators

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LITTLEWOOD-PALEY THEOREM FOR SCHRÖDINGER OPERATORS

SHIJUN ZHENG

ABSTRACT. Let H be a Schrödinger operator on $\mathbb{R}^n$. Under a polynomial decay condition for the kernel of its spectral operator, we show that the Besov spaces and Triebel-Lizorkin spaces associated with H are well defined. We further give a Littlewood-Paley characterization of L_p spaces in terms of dyadic functions of H. This generalizes and strengthens the previous result when the heat kernel of H satisfies certain upper Gaussian bound.

1. INTRODUCTION AND MAIN RESULTS

Recently the theory of function spaces associated with Schrödinger operators have been drawing attention in the area of harmonic analysis and PDEs [12, 2, 1, 14, 16, 9, 6, 8, 7, 5]. In [9, 6, 1, 14] it is proved that the Besov and Trieble-Lizorkin spaces associated with a Schrödinger operator are well defined, in some particular cases. In this note we aim to extend the result for general Schrödinger operators on $\mathbb{R}^n$. Furthermore we are interested in obtaining a Littlewood-Paley decomposition for the L_p spaces as well as Sobolev spaces using dyadic functions of H.

Let $H = -\Delta + V$ be a Schrödinger operator that is selfadjoint in $L_2(\mathbb{R}^n)$ with a real-valued potential function V. Then for a Borel measurable function ϕ, one can define the spectral operator $\phi(H)$ by functional calculus $\phi(H) = \int_{-\infty}^{\infty} \phi(\lambda) dE_\lambda$, where dE_λ is the spectral measure of H. The kernel of $\phi(H)$ is denoted $\phi(H)(x, y)$.

Let $\{\varphi_j\}_{j\in\mathbb{Z}} \subset C_0^\infty(\mathbb{R})$ be a smooth dyadic system satisfying the conditions (i) supp $\varphi_j \subset \{x : 2^{j-2} \le |x| \le 2^j\}$

(ii) $|\varphi_j^{(k)}(x)| \le c_k 2^{-kj}$, $\quad \forall j \in \mathbb{Z},\ k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$

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(iii)

$$\sum_{j=-\infty}^{\infty} \varphi_j(x) \approx c > 0\,, \qquad \forall x \neq 0\ .$$

Let $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$. The homogenous *Triebel-Lizorkin space* $\dot{F}_p^{\alpha,q}(H)$ is defined as the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with the quasi-norm

$$\|f\|_{\dot{F}_p^{\alpha,q}(H)} = \|\big(\sum_{j=-\infty}^{\infty} 2^{j\alpha q}|\varphi_j(H)f(\cdot)|^q\big)^{1/q}\|_p.$$

Similarly, if $0 < p \leq \infty$, $0 < q \leq \infty$, the homogeneous *Besov space* $\dot{B}_p^{\alpha,q}(H)$ is defined by the quasi-norm

$$\|f\|_{\dot{B}_p^{\alpha,q}(H)} = \big(\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\|\varphi_j(H)f\|_p^q\big)^{1/q}.$$

Throughout this note we assume H satisfies the following:

Assumption 1.1. *Let $\phi_j \in C_0^\infty(\mathbb{R})$ be as in condition (i), (ii). Then for every $N \in \mathbb{N}_0$ there exists a constant $c_N > 0$ such that for all $j \in \mathbb{Z}$*

$$|\phi_j(H)(x,y)| \leq c_N \frac{2^{nj/2}}{(1+2^{j/2}|x-y|)^N} \tag{1}$$

$$|\nabla_x \phi_j(H)(x,y)| \leq c_N \frac{2^{(n+1)j/2}}{(1+2^{j/2}|x-y|)^N}. \tag{2}$$

This is the case when H is the Hermite operator $-\Delta + |x|^2$, or more generally, whenever V is nonnegative and H satisfies the upper Gaussian bound for the heat kernel and its derivative (see Proposition 3.3). However, when the potential V is negative, such a heat kernel estimate is *not* available. Therefore it is necessary to consider a more general condition as given in Assumption 1.1.

Define the Peetre maximal function for H as: for $j \in \mathbb{Z}$, $s > 0$

$$\varphi_{j,s}^* f(x) = \sup_{t\in\mathbb{R}^n} \frac{|\varphi_j(H)f(t)|}{(1+2^{j/2}|x-t|)^s}\,,$$

and

$$\varphi_{j,s}^{**} f(x) = \sup_{t\in\mathbb{R}^n} \frac{|(\nabla_t \varphi_j(H)f)(t)|}{(1+2^{j/2}|x-t|)^s}\,.$$

The following theorem is a maximal characterization of the *homogeneous* spaces. By $\|\cdot\|_A \approx \|\cdot\|_B$ we mean equivalent norms.

Theorem 1.2. *Suppose H satisfies Assumption 1.1.*

a) If $0 < p \le \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$ and $s > n/p$, then

$$\|f\|_{\dot{B}_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha}\varphi_{j,s}^*(H)f\}\|_{\ell^q(L_p)} \ .$$

b) If $0 < p < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$ and $s > n/\min(p,q)$, then

$$\|f\|_{\dot{F}_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha}\varphi_{j,s}^*(H)f\}\|_{L_p(\ell^q)} \ .$$

It is well-known that such a characterization implies that any two dyadic systems satisfying (i), (ii), (iii) give rise to equivalent norms on $\dot{F}_p^{\alpha,q}(H)$ and $\dot{B}_p^{\alpha,q}(H)$. The analogous result also holds for the inhomogenous spaces $F_p^{\alpha,q}(H)$, $B_p^{\alpha,q}(H)$. However, the homogeneous spaces, which cover both high and low energy portion of H, are essential and more useful in proving Strichartz inequality for wave equations [16, 13]. This is one reason of our motivation.

Following the same idea in [14], using Calderón-Zygmund decomposition and Assumption 1.1 we show that $L_p(\mathbb{R}^n) = F_p^{0,2}(H)$ if $1 < p < \infty$. We thus obtain the Littlewood-Paley theorem for L_p spaces.

Theorem 1.3. *Suppose H satisfies Assumption 1.1. If $1 < p < \infty$, then*

$$\|f\|_{L_p(\mathbb{R}^n)} \approx \|\Big(\sum_{j=-\infty}^{\infty} |\varphi_j(H)f(\cdot)|^2\Big)^{1/2}\|_{L_p(\mathbb{R}^n)} \ .$$

Under additional condtion on V, e.g., $|\partial_x^k V(x)| \le c_k$, $|k| \le 2m_0 - 2$ for some $m_0 \in \mathbb{N}$, we can characterize the Sobolev spaces $H_p^{2s}(\mathbb{R}^n) = F_p^{s,2}(H)$, $1 < p < \infty$, $|s| \le m_0$ with equivalent norms

$$\|f\|_{H_p^{2s}(\mathbb{R}^n)} \approx \|\Big(\sum_{j=-\infty}^{\infty} 2^{2js}|\varphi_j(H)f(\cdot)|^2\Big)^{1/2}\|_{L_p(\mathbb{R}^n)} \ .$$

2. Proofs of Theorem 1.2 and Theorem 1.3

The proof of Theorem 1.2 is standard and follows from Bernstein type inequality (Lemma 2.1) and Peetre type maximal inequality (Lemma 2.2) for maximal functions.

Lemma 2.1. *For $s > 0$, there exists a constant $c_{n,s} > 0$ such that for all $j \in \mathbb{Z}$*

$$\varphi_{j,s}^{**}f(x) \le c_{n,s}2^{j/2}\varphi_{j,s}^*f(x), \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Similar to [20, 14] Lemma 2.1 can be easily proved using (2) with $N > n + s$ and the identity

$$\varphi_j(H)f(x) = \psi_j(H)\varphi_j(H)f,$$

where $\psi_j(x) = \psi(2^{-j}x)$ with $\psi \in C_0^\infty$, supp $\psi \subset \{\frac{1}{5} \le |x| \le \frac{5}{4}\}$ and $\psi(x) = 1$ on $\{\frac{1}{4} \le |x| \le 1\}$.

Let M denote the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\,dy \tag{3}$$

where the supreme is taken over all balls B in $\mathbb{R}^n$ centered at x.

Lemma 2.2. *Let $0 < r < \infty$ and $s = n/r$. Then for all $j \in \mathbb{Z}$*

$$\varphi_{j,s}^* f(x) \le c_{n,r}[M(|\varphi_j(H)f|^r)]^{1/r}(x), \qquad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{4}$$

Proof. Let $g(x) \in C^1(\mathbb{R}^n)$. As in [20, 1], the mean value theorem gives for $z_0 \in \mathbb{R}^n$, $\delta > 0$

$$|g(z_0)| \le \delta \sup_{|z-z_0|\le\delta} |\nabla g(z)| + c_{n,r}\delta^{-n/r}\left(\int_{|z-z_0|\le\delta} |g|^r dz\right)^{1/r}.$$

Put $g(z) = \varphi_j(H)f(x-z)$ to get

$$\frac{|\varphi_j(H)f(x-z)|}{(1+2^{j/2}|z|)^{n/r}} \le \delta \sup_{|u-z|\le\delta} \frac{(1+2^{j/2}|u|)^{n/r}|\nabla(\varphi_j(H)f)(x-u)|}{(1+2^{j/2}|z|)^{n/r}(1+2^{j/2}|u|)^{n/r}}$$

$$+c_{n,r}\delta^{-n/r}(1+2^{j/2}|z|)^{-n/r}(\int_{|u-z|\le\delta} |\varphi_j(H)f(x-u)|^r du)^{1/r}$$

$$\le \delta(1+2^{j/2}\delta)^{n/r}\varphi_{j,s}^{**}f(x) + c_{n,r}\delta^{-n/r}\big(\frac{|z|+\delta}{1+2^{j/2}|z|}\big)^{n/r}[M(|\varphi_j(H)f|^r)(x)]^{1/r}$$

$$\le c_{n,r}\delta(1+2^{j/2}\delta)^{n/r}2^{j/2}\varphi_{j,s}^*f(x) + c_{n,r}\delta^{-n/r}\big(\frac{|z|+\delta}{1+2^{j/2}|z|}\big)^{n/r}[M(|\varphi_j(H)f|^r)]^{1/r}(x)$$

$$\le c_{n,r}\epsilon(1+\epsilon)^{n/r}\varphi_{j,s}^*f(x) + c_{n,r}(1+\epsilon^{-1})^{n/r}[M(|\varphi_j(H)f|^r)]^{1/r}(x),$$

by setting $\delta = 2^{-j/2}\epsilon$, $\epsilon > 0$ and using Lemma 2.1. Finally, taking $\epsilon > 0$ sufficiently small establishes (4). □

Now Theorem 1.2 is a consequence of Lemma 2.2 and the following well-known lemma on Hardy-Littlewood maximal function by a standard argument; see [20] or [9, 14] for some simple details.

Lemma 2.3. *a) If $1 < p \le \infty$, then*

$$\|Mf\|_{L_p(\mathbb{R}^n)} \le C_p\|f\|_{L_p(\mathbb{R}^n)}. \tag{5}$$

b) If $1 < p < \infty$, $1 < q \le \infty$, then

$$\|\left(\sum_j |Mf_j|^q\right)^{1/q}\|_{L_p(\mathbb{R}^n)} \le C_{p,q}\|\left(\sum_j |f_j|^q\right)^{1/q}\|_{L_p(\mathbb{R}^n)}. \tag{6}$$

2.4. **Proof of Theorem 1.3.** From the proof of the identification of $F_p^{0,2}(H)$ spaces [14, Theorem 5.1] we observe that the estimates in (1), (2) imply

$$\|f\|_{F_p^{0,2}(H)} \approx \|f\|_{L_p}\ , \qquad 1 < p < \infty \tag{7}$$

by applying $L_p(\ell^2)$-valued Calderón-Zygmund decomposition. On the other hand, Theorem 1.2 suggests that

$$\|f\|_{F_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha}\varphi_j(H)f\}\|_{L_p(\ell^q)} \tag{8}$$

whenever $\{\varphi_j\}_{j\in\mathbb{Z}}$ is a dyadic system satisfying (i), (ii), (iii).

Combining (7) and (8) with $\alpha = 0$, $q = 2$ proves Theorem 1.3. □

Remark 2.5. *For $p = 1$, Dziubański and Zienkiewicz [7] recently obtained a characterization of Hardy space associated with H and showed that if a compactly supported positive potential V is in $L^{n/2+\epsilon}$, $n \geq 3$, then*

$$\|f\|_{\mathcal{H}^1} \approx \|wf\|_{H^1(\mathbb{R}^d)}\ ,$$

where $\mathcal{H}^1 = \{f \in L^1 : \sup_{t>0} |e^{-tH}f(\cdot)| \in L^1\}$ and the weight w is defined by $w(x) = \lim_{t\to\infty}\int_{\mathbb{R}^n} e^{-tH}(x,y)dy$. It would be very interesting to see whether one can give a Littlewood-Paley characterization of $\mathcal{H}^1$ in the sense of Theorem 1.3.

3. Potentials satisfying upper Gaussian bound

In this section we show that Assumption 1.1 is verified when H satisfies the upper Gaussian bound (10) for its heat kernel. We begin with a weighted L^1 inequality, which is an easy consequence of [11, Lemma 8] by a scaling argument.

Lemma 3.1. *(Hebisch) Suppose $V \geq 0$ and e^{-tH} satisfies*

$$0 \leq e^{-tH}(x,y) \leq c_n t^{-n/2} e^{-c|x-y|^2/t}\ , \qquad \forall t > 0. \tag{9}$$

If $s > (n+1)/2 + \beta$, $\beta \geq 0$ and $\operatorname{supp} g \subset [-10, 10]$, then

$$\sup_{j\in\mathbb{Z},\, y\in\mathbb{R}^n} \|g(2^{-j}H)(\cdot,y)\langle 2^{j/2}(\cdot - y)\rangle^\beta\|_{L^1(\mathbb{R}^n)} \leq c_n\|g\|_{H^s(\mathbb{R})}\ ,$$

where $\langle x\rangle := 1 + |x|$ and $\|\cdot\|_{H^s}$ denotes the usual Sobolev norm.

Remark 3.2. *It is known that (9) holds whenever $V \geq 0$ is locally integrable.*

Proposition 3.3. *Let $\alpha = 0, 1$. Suppose $V \geq 0$ and e^{-tH} satisfies the upper Gaussian bound*

$$|\nabla_x^\alpha e^{-tH}(x,y)| \leq c_n t^{-(n+\alpha)/2} e^{-c|x-y|^2/t}, \qquad \forall t > 0. \tag{10}$$

If $\{\varphi_j\}_{j\in\mathbb{Z}}$ is a dyadic system satisfying (i), (ii), then for each $N \geq 0$

$$|\nabla_x^\alpha \varphi_j(H)(x,y)| \leq c_N 2^{j(n+\alpha)/2}(1+2^{j/2}|x-y|)^{-N}, \qquad \forall j\,.$$

Proof. Write

$$\nabla_x^\alpha \varphi_j(H)(x,y) = \int_z \nabla_x^\alpha e^{-tH}(x,z)(e^{tH}\varphi_j(H))(z,y)dz.$$

By (10) we have

$$\begin{aligned}
&|\nabla_x^\alpha \varphi_j(H)(x,y)| \\
&\leq c_n t^{-(n+\alpha)/2} \int e^{-c|x-z|^2/t} \langle (x-z)/\sqrt{t}\rangle^N \langle (x-z)/\sqrt{t}\rangle^{-N} \langle (z-y)/\sqrt{t}\rangle^{-N} \\
&\qquad \cdot \langle (z-y)/\sqrt{t}\rangle^N |(e^{tH}\varphi_j(H))(z,y)|dz \\
&\leq c_n t^{-(n+\alpha)/2} \langle (x-y)/\sqrt{t}\rangle^{-N} \int \langle (z-y)/\sqrt{t}\rangle^N |(e^{tH}\varphi_j(H))(z,y)|dz.
\end{aligned}$$

Setting $t = t_j := 2^{-j}$, we see that $g_j(x) := e^{t_j x}\varphi_j(x)$ also satisfies conditions (i), (ii). Writing $g_j(x) = g_0(2^{-j}x)$, then supp $g_0 \subset \{\frac{1}{4} \leq |x| \leq 1\}$ and

$$\|g_0\|_{H^N(\mathbb{R})} \leq \|g_j(2^j x)\|_{C^N(\mathbb{R})} \leq c_N\,.$$

Thus an application of Lemma 3.1 with $g = g_0$, $\beta = N$ proves the proposition. $\square$

3.4. **Hermite operator $H = -\Delta + |x|^2$.** To verifies Assumption 1.1 it is sufficient to show e^{-tH} satisfies the upper Gaussian bound in (10), according to Proposition 3.3.

For $k \in \mathbb{N}_0$, let h_k be the k^{th} Hermite function with $\|h_k\|_{L_2(\mathbb{R})} = 1$ such that

$$(-\frac{d^2}{dx^2} + x^2)h_k = (2k+1)h_k\,.$$

Then $\{h_k\}_0^\infty$ forms a complete orthonormal system (ONS) in $L_2(\mathbb{R})$. In $L_2(\mathbb{R}^n)$, the ONS is given by $\Phi_k(x) := h_{k_1}\otimes\cdots\otimes h_{k_n}$, $k = (k_1,\ldots,k_n) \in \mathbb{N}_0^n$.

By Mehler's formula [18, Ch.4] or [19], the heat kernel has the expression

$$\begin{aligned} e^{-tH}(x,y) = &\sum_{k\in\mathbb{N}_0^n} e^{-t(n+2|k|)}\Phi_k(x)\Phi_k(y) \\ &= \frac{1}{(2\pi\sinh(2t))^{n/2}} e^{-\frac{1}{2}\coth(2t)(|x|^2+|y|^2)+\mathrm{cosech}(2t)x\cdot y} \end{aligned}$$

for all $t>0$, $x,y\in\mathbb{R}^n$.

It is easy to calculate to find that there exist constants $c,c'>0$, $0<c_0,c_1,c_0',c_1'<1$ and $t_0>1$ such that

$$p_{\frac{t}{2}}(x,y) \le c\begin{cases} t^{-n/2}e^{-c_0|x-y|^2/t} & t\le t_0 \\ e^{-nt/2}e^{-c_1|x-y|^2} & t>t_0 \end{cases}$$

$$|\nabla_x p_{\frac{t}{2}}(x,y)| \le c'\begin{cases} t^{-(n+1)/2}e^{-c_0'|x-y|^2/t} & t\le t_0 \\ e^{-nt/2}e^{-c_1'|x-y|^2} & t>t_0\,, \end{cases}$$

where $p_t(x,y) := e^{-tH}(x,y)$. Hence (10) holds.

Remark 3.5. *For the Hermite operator, the decay estimates similar to (1), (2) were previously obtained in* [10] *in one dimension and* [6] *in n-dimension. The latter used Heisenberg group method. Proposition 3.3 shows that using heat kernel estimate we can obtain a simpler proof.*

Remark 3.6. *When V is negative, the heat kernel estimate (9) is not available, espcecially in low dimensions $n=1,2$. but Assumption 1.1 still holds in the high energy case ($j\ge 0$) for certain short range potentials. A special example is the one dimensional Pöschl-Teller model $V(x) = -\nu(\nu+1)\,\mathrm{sech}^2 x$, $\nu\in\mathbb{N}$, cf.* [14]. *We will discuss the problem in more detail in* [23] *where V is assumed to have only polynomial decay at infinity.*

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